DETECTING TORSION IN SKEIN MODULES USING HOCHSCHILD HOMOLOGY

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ABSTRACT. Given a Heegaard splitting of a closed 3-manifold, the skein modules of the two handlebodies are modules over the skein algebra of their common boundary surface. The zeroth Hochschild homology of the skein algebra of a surface with coefficients in the tensor product of the skein modules of two handlebodies is interpreted as the skein module of the 3-manifold obtained by gluing the two handlebodies together along this surface. A spectral sequence associated to the Hochschild complex is constructed and conditions are given for the existence of algebraic torsion in the skein module of this 3-manifold.

1. INTRODUCTION

Skein modules were introduced independently by Przytycki [12] and Turaev [16] and have been an active topic of research since their introduction. In particular, skein modules underlie quantum invariants [10, 9] and are connected to the representation theory of the fundamental group of the manifold [3, 13].

The skein module is spanned by the equivalence classes of framed links in the 3-manifold. The skein module of the cylinder over a surface has a multiplication that comes from laying one framed link on top of the other. With this multiplication, the skein module of the cylinder over a surface is an algebra.

Given a Heegaard splitting $H_0 \cup_F H_1$ of a 3-manifold, the skein module of each handlebody $B_i = K(H_i)$ is a module over the skein algebra of the cylinder over the common boundary surface $A = K(F)$. If $H_0$ is glued to $F \times \{0\}$ and $H_1$ is glued to $F \times \{1\}$, then $B_0$ is a left $A$-module and $B_1$ is a right $A$-module. Hence $B_0 \otimes B_1$ is a bimodule over $A$.

We will use the interplay between the two handlebodies and their common boundary surface to compute the Hochschild homology of $A$ with coefficients in $B_0 \otimes B_1$. We will then construct a spectral sequence and show how it can be used to detect algebraic torsion in the skein module of the manifold.
2. Preliminaries

2.1. Skein Modules. Let $\mathcal{L}(M)$ denote the set of isotopy classes of framed links in $M$, including the empty link, $\phi$. Let $R = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials. Consider the free module $R\mathcal{L}(M)$ with basis $\mathcal{L}(M)$. Define $S(M)$ to be the smallest submodule of $R\mathcal{L}(M)$ containing all expressions of the form $\bigcirc - t \bigcirc - t^{-1} \bigcirc$ and $L \sqcup \bigcirc + (t^2 + t^{-2})L$, where $L$ is any framed link and where the framed links in each expression are identical outside the regions shown in the diagrams. The Kauffman bracket skein module $K(M)$ is the quotient $R\mathcal{L}(M)/S(M)$.

Because $K(M)$ is defined using local relations on framed links, two homeomorphic manifolds have isomorphic skein modules. Thus $K(M)$ is an invariant of the 3-manifold $M$.

2.2. Heegaard Splittings. Let $M$ be a closed, orientable, connected 3-manifold. Then for some non-negative integer $g$ there exist genus $g$ handlebodies $H_0$ and $H_1$ such that $H_0 \cap H_1 = \partial H_0 = \partial H_1 = F$ is a closed, orientable, connected genus $g$ surface and $H_0 \cup H_1 = M$. We call these two handlebodies a Heegaard splitting of the manifold. A simple proof of the existence of Heegaard splittings using a triangulation of the manifold can be found in Rolfsen [14]. A given manifold may have many different Heegaard splittings.

Note that we can take a neighborhood of the surface $F$ and think of the Heegaard splitting as breaking the manifold into $H_0$, $F \times [0, 1]$, and $H_1$, where $H_0$ is glued to $F \times \{0\}$ by the identity map and $H_1$ is glued to $F \times \{1\}$ by a gluing map $f$. We will model Heegaard splittings in this way, and we will be interested in the properties of the gluing map $f$.

2.3. Hochschild Homology. Hochschild homology is a functor that associates an ordered collection of $R$-modules to an $R$-algebra $A$ and an $A$-bimodule $B$. The Hochschild chain complex has chains $C_n$ given by

$$C_n = C_n(A; B) = B \otimes (A^{\otimes n}) = B \otimes A \otimes A \otimes \ldots \otimes A \quad \text{$n$ times}$$
for \( n \geq 0 \) and \( C_n = 0 \) for \( n < 0 \). The Hochschild boundary map \( d_n : C_n \to C_{n-1} \) is given by

\[
d_n(b \otimes a_1 \otimes \ldots \otimes a_n) = ba_1 \otimes a_2 \otimes a_3 \otimes \ldots \otimes a_n
\]

\[
- b \otimes a_1a_2 \otimes a_3 \otimes \ldots \otimes a_n
\]

\[
+ b \otimes a_1 \otimes a_2a_3 \otimes \ldots \otimes a_n
\]

\[
+ \ldots + (-1)^{n-1}b \otimes a_1 \otimes \ldots \otimes a_{n-1}a_n
\]

\[
+ (-1)^n a_nb \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1}
\]

where the products \( a_ia_{i+1} \) take place in the algebra \( A \) and the products \( ba_1 \) and \( a_nb \) come from the respective right and left actions on \( B \) by \( A \). The Hochschild homology of \( A \) with coefficients in \( B \) is the homology of the Hochschild complex and is denoted \( HH_*(A; B) \). If \( B = A \) we will denote \( HH_*(A; A) \) by \( HH_*(A) \).

### 3. The Hochschild Homology of a Heegaard Splitting

#### 3.1. The skein module of a Heegaard splitting.

**Lemma 1.** (Hoste-Przytycki [7]) Consider the manifold \( F \times [0,1] \) where \( F \) is a surface. Let \( \alpha \) be a simple closed curve on \( F \times \{0\} \) (or \( F \times \{1\} \)). Let \( H \) be the manifold obtained by attaching a 2-handle to \( \alpha \). Let \( I \) be the submodule of \( K(F) \) generated by relations of the form \( \{s - h(s)\} \) where \( s \in K(F) \) is a skein and \( h(s) \) is the skein \( s \) modified by a handleslide across \( \alpha \). Then \( K(H) = K(F)/I \).

The lemma above generalizes to the case where the manifold is obtained by attaching more than one 2-handle to \( F \times [0,1] \). Say we attach a 2-handle to \( F \times \{0\} \) along \( \alpha \) and attach another 2-handle to \( F \times \{1\} \) along \( \beta \). Call the resulting manifold \( M \). If \( I \) is the submodule of \( K(F) \) generated by handleslides along \( \alpha \) and \( J \) is the submodule of \( K(F) \) generated by handleslides along \( \beta \), then \( K(M) = K(F)/(I + J) \). We can use this result and a property of the tensor product to get the following proposition.

**Proposition 1.** (discussed by Frohman-Gelca in [5]) Let \( M \) be a closed, connected, oriented 3-manifold with Heegaard splitting \( M = H_0 \cup H_1 \), \( F = H_0 \cap H_1 \). Then \( K(M) = K(H_1) \otimes_{K(F)} K(H_0) \).

**Proof.** \( H_0 \) is obtained from \( F \times [0,1] \) by attaching 2-handles to \( F \times \{0\} \) along attaching curves \( \alpha_k \). Likewise, \( H_1 \) is obtained from \( F \times \{1\} \) by attaching 2-handles to along curves \( \beta_n \). For \( i \in \{0,1\} \), let \( S_i \) be the submodule of \( K(F) \) generated by handleslides across the \( \alpha_k \) or across the \( \beta_n \), respectively. We can apply Lemma 1 to each \( H_i \) and to \( M \). Then we have \( K(H_i) = K(F)/S_i \) and \( K(M) = K(F)/(S_1 + S_0) \).
We know that $A/I \otimes_A B \cong B/(IB)$ from homological algebra, see Osborne [11, Proposition 2.2]. Let $A = K(F)$. Consider $A/S_1 \otimes_A A/S_0 \cong \frac{A/S_0}{S_1(A/S_0)}$. An element from $S_1(A/S_0)$ looks like $sa + S_0$ where $s \in S_1$ and $a \in A$. An element of $\frac{A/S_0}{S_1(A/S_0)}$ looks like $(a' + S_0) + (sa + S_0)$. Recall that the empty skein $\phi$ is the multiplicative identity in $A$, thus $sa$ runs over all of $S_1$ and so $(a' + S_0) + (sa + S_0) = a' + (S_1 + S_0)$. Thus $K(H_1 \otimes_{K(F)} K(H_0) \cong A/S_1 \otimes_A A/S_0 \cong \frac{A/S_0}{S_1(A/S_0)} \cong A/(S_1 + S_0) \cong K(M)$.

\[\square\]

3.2. Connection with character varieties. Another interesting and useful property of the skein module $K(M)$ comes when we specialize at $t = -1$. The coordinate ring of the $SL_2(\mathbb{C})$-characters on $\pi_1(M)$ is a quotient of this specialization. This approach has been developed by Bullock [3] and also by Przytycki and Sikora [13].

Denote the specialization of $K(M)$ at $t = -1$ by $K_{-1}(M)$ and denote the space of $SL_2(\mathbb{C})$-characters by $X(M)$. Let $\mathbb{C}^{X(M)}$ denote the algebra of functions on $X(M)$ and let $R(M)$ denote the coordinate ring of $X(M)$.

By a theorem of Culler and Shalen [4], $X(M)$ is an affine algebraic set. Hence one can consider the ring of polynomial functions on $X(M)$. This ring of polynomial functions is called the coordinate ring of $X(M)$. Indeed, Culler and Shalen show that the coordinate ring is finitely generated.

An oriented knot in $M$ determines a conjugacy class in $\pi_1(M)$ and thus an oriented knot $L$ defines a function $\varphi_L : X(M) \to \mathbb{C}$ by $\varphi_L(\chi) = \chi(L) = \text{tr}(\rho(L))$ where $\chi_L$ is the character induced by the representation $\rho$ and the knot $L$ is seen as an element of $\pi_1(M)$. Since $\text{tr}(A) = \text{tr}(A^{-1})$ for any matrix $A$, the particular orientation on the knot $L$ is irrelevant.

Let $\mathcal{L}(M)$ denote the vector space of framed links in $M$. Define a function $\Phi : \mathcal{L}(M) \to \mathbb{C}^{X(M)}$ by $\Phi(L) = -\varphi_L$ for a knot $L$ and $\Phi(L) = \prod_{i}(-\varphi_{L_i})$ for a link $L$ with components $L_i$.

**Lemma 2.** (Bullock [3]) The map $\Phi$ descends to a map $\Phi : K_{-1}(M) \to \mathbb{C}^{X(M)}$. Its image is the coordinate ring $R(M) \subset \mathbb{C}^{X(M)}$ and its kernel is the nilradical of $K_{-1}(M)$.

The proof that the map descends follows from the observation that the skein relation maps to the $SL_2(\mathbb{C})$ trace identity $\text{tr}(AB) + \text{tr}(AB^{-1}) = \text{tr}(A)\text{tr}(B)$. 
Przytycki and Sikora [13] have shown that the nilradical of $K^{-1}(M)$ is trivial for surfaces and handlebodies. Thus for surfaces and handlebodies $\Phi$ is an isomorphism between the specialized skein module and the coordinate ring of the character variety.

**Example 1.** As an example, let's look at the 3-manifold $M = T^2 \times [0, 1]$. We know that $\pi_1(M) = \langle \ell, m \mid \ell m t^{-1} m^{-1} = 1 \rangle$. The coordinate ring $R(M)$ is generated by $x = -\text{tr}(\rho(m))$, $y = -\text{tr}(\rho(\ell))$, and $z = -\text{tr}(\rho(\ell m))$, and it has one relation induced by $\text{tr}(\ell m t^{-1} m^{-1}) = 2$. Hence

$$R(M) \cong \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2 + xyz - 4).$$

### 3.3. Using Hochschild Homology.

Now we can use the connection between the specialized skein module and the coordinate ring in the context of a Heegaard splitting.

Let $M$ be a 3-manifold with Heegaard splitting $M = H_0 \cup F \times [0, 1] \cup H_1$ and with gluing maps $f_0 : H_0 \to F \times \{0\}$ and $f_1 : H_1 \to F \times \{1\}$. We will take $f_0$ to be the identity, hence the structure of the manifold is described by $f_1$.

The action of $K(F)$ on $K(H_i)$ is given by pushing the skeins from $F \times I$ into $H_i$ using the inverse of the $f_i$ gluing map. The action of $\alpha, \beta \in K(F)$ on $h_0 \in K(H_0)$ is a left action, $(\alpha \beta) * h_0 = \alpha * (\beta * h_0)$, as shown in Figure 1. The action of $\alpha, \beta \in K(F)$ on $h_1 \in K(H_1)$ is a right action, $h_1 * (\alpha \beta) = (h_1 * \alpha) * \beta$, as shown in Figure 2. For $R = \mathbb{Z}[t, t^{-1}]$, we know that $A = K(F)$ is an algebra over $R$. Also, $B_0 = K(H_0)$, and $B_1 = K(H_1)$ are modules over $R$. Since $B_0$ is a left $A$-module and $B_1$ is a right $A$-module, the tensor product $B = B_0 \otimes_R B_1$ is a bimodule over $A$. We will use the unspecified tensor $\otimes$ to denote $\otimes_R$. 

![Figure 1](image1.png) 

**Figure 1.** $(\alpha \beta) * h_0 = \alpha * (\beta * h_0)$ defines a left action
Now we look at $HH_\ast(A; B_0 \otimes B_1)$. With the choice of $B = B_0 \otimes B_1$, the chains become

$$C_n(A; B_0 \otimes B_1) = (B_0 \otimes B_1) \otimes (A^{\otimes n})$$

for $n \geq 0$ and $C_n = 0$ for $n < 0$. Rearrange the terms in the tensor product so that the $C_n$ become

$$C_n(A; B_0 \otimes B_1) = B_1 \otimes A \otimes A \otimes \ldots \otimes A \otimes B_0.$$  

Then $d_n$ is

$$d_n(b_1 \otimes a_1 \otimes \ldots \otimes a_n \otimes b_0) = b_1 a_1 \otimes a_2 \otimes a_3 \otimes \ldots \otimes a_n \otimes b_0$$

$$- b_1 \otimes a_1 a_2 \otimes a_3 \otimes \ldots \otimes a_n \otimes b_0$$

$$+ \ldots + (-1)^n b_1 \otimes a_1 \otimes \ldots \otimes a_n b_0.$$  

Thus, $d_n : B_1 \otimes (A^{\otimes n}) \otimes B_0 \rightarrow B_1 \otimes (A^{\otimes(n-1)}) \otimes B_0$.

Notice that without the $B_1$, the sequence

$$\cdots \rightarrow A \otimes A \otimes A \otimes B_0 \rightarrow A \otimes A \otimes B_0 \rightarrow A \otimes B_0 \rightarrow B_0 \rightarrow 0$$

is a free (hence projective) resolution of $B_0$. If we delete $B_0$ from this complex, tensor over $A$ on the left by $B_1$ and compute the homology, we get $\Tor^A_i(B_1, B_0)$. That is, $\Tor^A_i(B_1, B_0)$ is the homology of the following complex

$$\cdots \rightarrow B_1 \otimes A \otimes A \otimes A \otimes B_0 \rightarrow B_1 \otimes A \otimes A \otimes B_0 \rightarrow B_1 \otimes A \otimes B_0 \rightarrow B_1 \otimes B_0 \rightarrow 0.$$  

Since $B_1 \otimes A = B_1$, the complex becomes

$$\cdots \rightarrow B_1 \otimes A \otimes A \otimes B_0 \rightarrow B_1 \otimes A \otimes B_0 \rightarrow B_1 \otimes B_0 \rightarrow 0.$$  

Therefore the Tor complex is exactly the same as the Hochschild complex, that is,

$$HH_i(A; B_0 \otimes B_1) = \Tor^A_i(B_1, B_0).$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{\(h_1 \ast (\alpha \beta) = (h_1 \ast \alpha) \ast \beta\) defines a right action}
\end{figure}
**Lemma 3.** The zeroth Hochschild homology of a Heegaard splitting is the skein module of the 3-manifold $M$.

**Proof.** Tor$_0$ corresponds to $\otimes$, thus Tor$_0^A(B_1, B_0) = B_1 \otimes_A B_0$. We know from Proposition 1 that $K(M) = K(H_1) \otimes_{K(F)} K(H_0)$, thus $K(M) = B_1 \otimes_A B_0 = \text{Tor}_0^A(B_1, B_0) = \text{HH}_0(A; B_0 \otimes B_1)$.

4. A Spectral Sequence to Detect Torsion

Next we use a filtration on $R$, $A$, $B_0$, and $B_1$ to get a spectral sequence and search for $(1 + t)$-torsion in $K(M)$. We will follow a process used by Brylinski in [2] to study Poisson manifolds.

The ring $R = \mathbb{Z}[t, t^{-1}]$ of Laurent polynomials has a filtration by the ideals corresponding to powers of $(1 + t)$.

$$\cdots \subset (1 + t)^3 R \subset (1 + t)^2 R \subset (1 + t)R \subset R$$

This is a decreasing filtration. By manipulating the indices, we can use this filtration to get an increasing filtration. In particular, if $\mathcal{F}_s(R) = (1 + t)^{-s} R$ for $s \leq 0$ and $\mathcal{F}_s(R) = R$ for $s > 0$, then $\mathcal{F}$ is an increasing filtration on $R$. This filtration extends to the $R$-modules $A = K(F)$ and $B_i = K(H_i)$ by

$$\cdots \subset (1 + t)^3 A \subset (1 + t)^2 A \subset (1 + t)A \subset A$$

and

$$\cdots \subset (1 + t)^3 B_i \subset (1 + t)^2 B_i \subset (1 + t)B_i \subset B_i.$$ 

It also extends to the Hochschild complex $C_n = C_n(A; B_0 \otimes B_1)$ by the following from Brylinski [2].

$$\mathcal{F}_s(C_n) = \mathcal{F}_s(B_1 \otimes (A^\otimes n) \otimes B_0)$$

$$= \sum_{s_0 + \cdots + s_{n+1} \leq s} (\mathcal{F}_{s_0}(B_1) \otimes \mathcal{F}_{s_1}(A) \otimes \cdots \otimes \mathcal{F}_{s_n}(A) \otimes \mathcal{F}_{s_{n+1}}(B_0))$$

$$= \sum_{\sum s_i \leq s} (1 + t)^{-\sum s_i} (B_1 \otimes (A^\otimes n) \otimes B_0)$$

Now we create a spectral sequence $\{E_r\}$ beginning at the $E^0$ level with

$$E^0_{p,q} = \mathcal{F}_p(C_{p+q})/\mathcal{F}_{p-1}(C_{p+q})$$

and the $E^0$ level boundaries $\Delta^0_n$ are induced by the Hochschild boundary map $d_n$ as described in Equation 1.

We move from the $E^0$ level to the $E^1$ level by taking homology and letting the new boundary map $\Delta^1_n$ be the connecting homomorphism.
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induced by the short exact sequence
\[ 0 \to \mathcal{F}_{p-1}(C_n) \to \mathcal{F}_p(C_n) \to \mathcal{F}_{p-1}(C_n) \to 0. \]

In general, \( E_{p,q}^r = H(F_{p,q}^{r-1}) = \ker(\Delta^{r-1})/ \text{im}(\Delta^{r-1}) \) and we set
\[ E_{p,q}^\infty = \lim_{r \to \infty} E_{p,q}^r. \]

The terms at all levels are only nonzero in the second quadrant above the line \( q = -p \). The \( E^0 \) level is shown in Figure 3. We will be particularly concerned with the terms \( E_{p,-p}^r \) along the lower diagonal.

\[ \begin{align*}
(1 + t)^3(B_1 \otimes B_0) \\
(1 + t)^4(B_1 \otimes B_0)
\end{align*} \]

\[ \begin{align*}
B_1 \otimes A^3 \otimes B_0 \\
(1 + t)(B_1 \otimes A^3 \otimes B_0)
\end{align*} \]

\[ \begin{align*}
(1 + t)^2(B_1 \otimes B_0) \\
(1 + t)^3(B_1 \otimes B_0)
\end{align*} \]

\[ \begin{align*}
B_1 \otimes A \otimes A \otimes B_0 \\
(1 + t)(B_1 \otimes A \otimes A \otimes B_0)
\end{align*} \]

\[ \begin{align*}
(1 + t)(B_1 \otimes B_0) \\
(1 + t)^2(B_1 \otimes B_0)
\end{align*} \]

\[ \begin{align*}
B_1 \otimes A \otimes B_0 \\
(1 + t)(B_1 \otimes A \otimes B_0)
\end{align*} \]

\[ \begin{align*}
B_1 \otimes B_0 \\
(1 + t)(B_1 \otimes B_0)
\end{align*} \]

\[ \begin{align*}
(1 + t)^3(B_1 \otimes B_0)
\end{align*} \]

\[ \begin{align*}
B_1 \otimes A^3 \otimes B_0
\end{align*} \]

\[ \begin{align*}
B_1 \otimes A \otimes B_0
\end{align*} \]

\[ \begin{align*}
B_1 \otimes B_0
\end{align*} \]

Figure 3. The \( E^0 \) level of the spectral sequence

Lemma 4. Every nonzero term of the \( E^0 \) level of the spectral sequence above is isomorphic to the tensor product of specialized skein modules. Namely, for \((p, q)\) with \( q \geq 0 \) and \(-q \leq p \leq 0\), we have
\[ E_{p,q}^0 \cong K_{-1}(H_1) \otimes (K_{-1}(F))^{\otimes (p+q)} \otimes K_{-1}(H_0) \]
Proof. By definition, the \((p, q)\) term of the \(E^0\) level is
\[
E_{p,q}^0 = \mathcal{F}_p(C_{p+q})/\mathcal{F}_{p-1}(C_{p+q})
\]
\[
= \sum_{k=-p}^{\infty} (1+t)^k(B_1 \otimes A^{\otimes(p+q)} \otimes B_0)
\]
and the boundary maps of the \(E^0\) level are the Hochschild boundary maps. Consider the complex in the column along the \(q\)-axis in Figure 3. These terms are the original Hochschild chains \(C_q = B_1 \otimes (A^{\otimes q}) \otimes B_0\) quotiented by the terms that have a factor of \((1+t)\) in their coefficients. Modding out by the ideal generated by \((1+t)\) is the same as setting \((1+t) = 0\) or simply evaluating the polynomials at \(t = -1\). Since evaluating at \(t = -1\) yields the specialized skink module, we have, for example, \(K(F)/(1+t)K(F) \cong K_{-1}(F)\). Recall that \(A = K(F)\), \(B_0 = K(H_0)\), and \(B_1 = K(H_1)\), so \(A/(1+t)A \cong K_{-1}(F)\), \(B_0/(1+t)B_0 \cong K_{-1}(H_0)\), and \(B_1/(1+t)B_1 \cong K_{-1}(H_1)\).

These quotients are consistent with the tensor product, thus,
\[
\frac{B_1 \otimes (A^{\otimes n}) \otimes B_0}{(1+t)(B_1 \otimes (A^{\otimes n}) \otimes B_0)} \cong K_{-1}(H_1) \otimes (K_{-1}(F))^{\otimes n} \otimes K_{-1}(H_0).
\]
In addition, for \(C_n = B_1 \otimes (A^{\otimes n}) \otimes B_0\) there is a natural map from \(C_n\) to \((1+t)^{-p}C_n\) given by multiplication by \((1+t)^{-p}\). Each of the skink modules \(A, B_0,\) and \(B_1\) is free on simple diagrams, so multiplying by \((1+t)^{-p}\) induces an isomorphism
\[
\frac{C_n}{(1+t)C_n} \cong \frac{(1+t)^{-p}C_n}{(1+t)^{-p+1}C_n}
\]
and therefore
\[
E_{p,q}^0 \cong \frac{(1+t)^{-p}(B_1 \otimes A^{\otimes(p+q)} \otimes B_0)}{(1+t)^{-p+1}(B_1 \otimes A^{\otimes(p+q)} \otimes B_0)}
\]
\[
\cong \frac{(B_1 \otimes A^{\otimes(p+q)} \otimes B_0)}{(1+t)(B_1 \otimes A^{\otimes(p+q)} \otimes B_0)}
\]
\[
\cong K_{-1}(H_1) \otimes (K_{-1}(F))^{\otimes(p+q)} \otimes K_{-1}(H_0)
\]
as desired. \(\square\)

To move from the \(E^0\) level to the \(E^1\) level in the spectral sequence, we take the homology of the vertical \(E^0\) level complexes. Since these
complexes are Hochschild complexes, we get the Hochschild homology. From the proof of Lemma 4, we know that

$$\frac{C_n}{(1 + t)C_n} \cong \frac{(1 + t)^{-p}C_n}{(1 + t)^{-p+1}C_n}.$$ 

It is enough, then, to focus our attention on the complex along the $q$-axis in Figure 3. Namely,

$$E^1_{0,q} = HH_q \left( \frac{A}{(1 + t)A}; \frac{B_0}{(1 + t)B_0} \otimes \frac{B_1}{(1 + t)B_1} \right)$$

and in general

$$(2) \quad E^1_{p,q} = HH_{p+q} \left( \frac{A}{(1 + t)A}; \frac{B_0}{(1 + t)B_0} \otimes \frac{B_1}{(1 + t)B_1} \right).$$

The boundary maps for the $E^1$ level are the induced connecting homomorphisms $\Delta^1_n : HH_n \to HH_{n-1}$. The $E^1$ level is shown in Figure 4.

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**Figure 4.** The $E^1$ level of the spectral sequence.
The connecting homomorphisms in this context are essentially the same as the boundary map, we are just considering the image in a different quotient space. For example, a 1-cycle in $E_{0,1}^0$ is an element $\alpha \in (B_1 \otimes A \otimes B_0)/(1 + t)(B_1 \otimes A \otimes B_0)$ that maps to zero in $(B_1 \otimes B_0)/(1 + t)(B_1 \otimes B_0)$, so $d_1(\alpha)$ is divisible by $(1 + t)$. The connecting homomorphism $\Delta$ is defined by $\Delta(\alpha) = d_1(\alpha)$, then consider $d_1(\alpha)$ in $(1 + t)(B_1 \otimes B_0)/(1 + t)^2(B_1 \otimes B_0)$. For example, $\Delta(\alpha)$ will be zero if $d_1(\alpha)$ is also divisible by $(1 + t)^2$.

Thus we have the following lemma.

**Lemma 5.** $\Delta_1^1 : HH_1 \to HH_0$ will be the zero map if every element whose boundary is divisible by $(1 + t)$ also has its boundary divisible by $(1 + t)^2$. 

When we move to the next level, we will be looking at something whose boundary is divisible by $(1 + t)^2$ and it will be zero under the next connecting homomorphism if its boundary is also divisible by $(1 + t)^3$.

Thus, all the connecting homomorphisms will be the zero map if every element whose boundary is divisible by $(1 + t)$ also has a boundary that is divisible by $(1 + t)^r$ for all $r$. This will happen if the boundary in question is *exactly* zero, not just some polynomial divisible by $(1 + t)$.

Again, consider Figure 4 and notice that these $\Delta$ maps are horizontal and pointing along the negative $x$ axis. Recall from Lemma 4 that the filtered complex (the $E^0$ level) can be seen as a complex of specialized skein modules. Thus the $E^1_{p,-p}$ terms are just the Hochschild homology of these specialized skein modules. Namely,

$$E^1_{p,-p} = HH_0 \left( \frac{A}{(1 + t)A}; \frac{(B_0 \otimes B_1)}{(1 + t)(B_0 \otimes B_1)} \right) = HH_0 (K_{-1}(F); K_{-1}(H_0) \otimes K_{-1}(H_1)) = K_{-1}(H_1) \otimes_{K_{-1}(F)} K_{-1}(H_0) = K_{-1}(M).$$

5. **LOOKING FOR TORSION**

We have constructed a spectral sequence from the filtered Hochschild complex. We now want to use this construction to look for $(1 + t)$-torsion in the skein module $K(M)$.

**Definition 1.** A module $X$ over a ring $R$ has torsion if there exist nonzero elements $r \in R$ and $x \in X$ such that $rx = 0$.

**Definition 2.** Let $M$ be a manifold. Recall that $K(M)$, the skein module of $M$, is a module over the Laurent polynomials $R = \mathbb{Z}[t, t^{-1}]$. 
The filtration $\mathcal{F}_s(R) = (1 + t)^{-s}R$ for $s \leq 0$ extends to $K(M)$ as
\[ \cdots \subset (1 + t)^{3}K(M) \subset (1 + t)^{2}K(M) \subset (1 + t)K(M) \subset K(M). \]
The quotients $K(M)/(1 + t)^n K(M)$ together with the projections $\theta_n : K(M)/(1 + t)^n K(M) \to K(M)/(1 + t)^{n-1} K(M)$ form an injective system. The completion of $K(M)$ is the inverse limit of this system,
\[ \overline{K(M)} = \lim_{\leftarrow} K(M)/(1 + t)^n K(M). \]
The completion can also be seen as the module of all sequences $\{a_n\}_{n=0}^{\infty}$ with $a_n \in K(M)/(1 + t)^n K(M)$ and $\theta_n(a_n) = a_{n-1}$. There is a homomorphism $\varphi : K(M) \to \overline{K(M)}$ where $\varphi(\alpha)$ is the constant sequence $\{\alpha\}_{n=0}^{\infty}$. Note that the kernel of this homomorphism is
\[ \text{Ker}(\varphi) = \bigcap_{n=0}^{\infty} (1 + t)^n K(M). \]

For more information on the definition above and properties of the completion, see Atiyah and MacDonald [1, Chapter 10].

**Theorem 1.** If the $\Delta^r_p : E^r_{p+1,-p} \to E^r_{p,-p}$ maps are identically zero at every level in the spectral sequence, then there is no $(1 + t)$-torsion in $K(M)$.

**Proof.** We focus our attention in the spectral sequence to the terms that lie along the lower diagonal (the line $q = -p$). These are the terms $E^r_{p,-p}$ for $p \leq 0$.

At the $E^1$ level, from Equation 2 we know that these terms have the form
\[ E^1_{p,-p} = HH_0 \left( \frac{A}{(1 + t)A} \otimes \frac{B_0}{(1 + t)B_0} \otimes \frac{B_1}{(1 + t)B_1} \right) \]
so these terms are the zeroth Hochschild homology modules of the various filtered complexes.

The maps $\Delta^1_1 : E^1_{p+1,-p} \to E^1_{p,-p}$ are zero maps, thus
\[ E^2_{p,-p} = \frac{\text{ker}(E^1_{p,-p} \to 0)}{\text{im}(\Delta^1_1)} = E^1_{p,-p}. \]
That is, the term at position $(p, -p)$ remains unchanged when we move from the $E^1$ level to the $E^2$ level.

The argument is the same for any $r$. The maps $\Delta^r_1 : E^r_{p+1,-p-r+1} \to E^r_{p,-p}$ are zero maps, thus
\[ E^{r+1}_{p,-p} = \frac{\text{ker}(E^r_{p,-p} \to 0)}{\text{im}(\Delta^r_1)} = E^r_{p,-p}. \]
Figure 5. The $E^\infty$ level of the spectral sequence

Thus after the $E^1$ level, the terms $E^r_{p,-p}$ along the lower diagonal are always just the zeroth Hochschild homology of the filtered complexes, which, from Equation 3, we know to be isomorphic to the specialized skein module of $M$. Thus, in the limit we have

$$E_{p,-p}^\infty = \lim_{r \to \infty} E^r_{p,-p} = E^1_{p,-p} \cong K_{-1}(M).$$

The $E^\infty$ level of the spectral sequence is shown in Figure 5. Each term along the lower diagonal is isomorphic to $K_{-1}(M)$. Specifically,

$$E_{p,-p}^\infty \cong \frac{(1+t)^{-p}K(M)}{(1+t)^{-p+1}K(M)},$$

and the map

$$\mu_n : \frac{K(M)}{(1+t)K(M)} \to \frac{(1+t)^nK(M)}{(1+t)^{n+1}K(M)}$$

given as multiplication by $(1+t)^n$ is an isomorphism for all $n \geq 1$.

Suppose there exists an $\alpha \in K(M)$ such that $(1+t)\alpha = 0$. Then $\mu_1(\alpha) = 0$. Since $\mu_1$ is an isomorphism, $\alpha$ must be zero in $K(M)/(1+
t)K(M). Hence $\alpha$ is divisible by $(1 + t)$. Let $\alpha_1 = \alpha/(1 + t)$. Then 
$\mu_2(\alpha_1) = (1 + t)^2 \alpha = 0$. Since $\mu_2$ is an isomorphism, $\alpha_1$ must be zero 
in $K(M)/(1 + t)K(M)$. Hence $\alpha_1$ is divisible by $(1 + t)$ and so $\alpha$ is 
divisible by $(1 + t)^2$. An induction argument shows that $\alpha$ is divisible 
by $(1 + t)^n$ for all $n$. Thus $\alpha \in \cap_{n=0}^{\infty}(1 + t)^n K(M)$ and so $\alpha = 0$ in the 
completion $\overline{K(M)}$. Therefore there is no $(1 + t)$-torsion in $\overline{K(M)}$.

The motivation for starting the search for torsion by looking for 
$(1 + t)$-torsion stems from the result of Hoste and Przytycki [8] about 
the skein module of $S^1 \times S^2$. Namely, they show that

$$K(S^1 \times S^2) \cong R \oplus \bigoplus_{i=1}^{\infty} R/(1 - t^{2i+4})$$

where $R = \mathbb{Z}[t, t^{-1}]$. Since $t = -1$ is a root of each of the $(1 - t^{2i+4})$ 
polynomials, all of the torsion in $K(S^1 \times S^2)$ can be interpreted as 
$(1 + t)$-torsion. It is natural to ask if $(1 + t)$-torsion is the only kind of 
torsion that a skein module can have. We conclude this section with 
the conjecture that $(1 + t)$-torsion is indeed the only kind of torsion.

If true, this conjecture would make the conclusions of Theorem 1 more 
general.

**Conjecture 1.** The absence of $(1 + t)$-torsion in $\overline{K(M)}$ implies the 
absence of torsion in $K(M)$.

6. Examples

The existence of torsion in $K(S^1 \times S^2)$ was shown by Hoste and 
Przytycki in [8]. The absence of torsion in the skein module of each of 
the other lens spaces was shown by them in [7]. Below we show an 
alternate way to see $(1 + t)$-torsion in $K(S^1 \times S^2)$ and the absence 
of $(1 + t)$-torsion in $K(L(2, 1))$.

6.1. The space $S^1 \times S^2$. Take the genus one Heegaard splitting $H_1 \cup f_1$ 
$T^2 \times I \cup f_0 H_0$ for $S^1 \times S^2$ where $f_0 : \partial H_0 \to T^2 \times \{0\}$ is the identity 
map and $f_1 : \partial H_1 \to T^2 \times \{1\}$ is

$$f_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Consider the element $\alpha \in \text{Tor}_1^{K^{-1}(T^2)}(K_{-1}(H_1), K_{-1}(H_0))$ given by

$$\alpha = \phi \otimes (p(t) \ast y - q(t) \ast z) \otimes \phi = \phi \otimes \left( p(t) \ast \overline{\bigotimes} - q(t) \ast \overline{\bigotimes} \right) \otimes \phi$$
with \( p(-1) = q(-1) = 1 \). Now consider \( \Delta_1^1(\alpha) \) which is given by
\[
\Delta_1^1(\alpha) = (p(t) + t^3 q(t)) \otimes \phi
\]
Since both \((p(t) + t^3 q(t))\) and \((p(t) + t^{-3} q(t))\) are divisible by \((1 + t)\), the element \( \alpha \) is a cycle and thus represents a nontrivial element in \( \text{Tor}_1 \). However, \((p(t) + t^3 q(t))\) and \((p(t) + t^{-3} q(t))\) are not divisible by \((1 + t)^2\), so \( \Delta_1^1(\alpha) \neq 0 \).

Now Theorem 1 does not apply, but if we mimic the proof of Theorem 1 for this example we will see that at the \( E^2 \) level of the spectral sequence the term \( E_{1,1}^2 \) is no longer isomorphic to \( E_{1,1}^1 \). Instead it is the nontrivial quotient of \( E_{1,1}^1 \) by the image of \( E_{0,1}^1 \) under \( \Delta_1^1 \). Compare Figure 4. Thus at the \( E^\infty \) level, the map
\[
\frac{K(S^1 \times S^2)}{(1 + t)K(S^1 \times S^2)} \to \frac{(1 + t)K(S^1 \times S^2)}{(1 + t)^2K(S^1 \times S^2)}
\]
has nontrivial kernel. Thus there exists an element \( \beta \in K(S^1 \times S^2) \) that is not divisible by \((1 + t)\) such that \((1 + t)\beta \) is divisible by \((1 + t)^2\). Thus \((1 + t)\beta = 0\) and \((1 + t)\)-torsion exists in \( K(S^1 \times S^2) \).

6.2. The \( L(2,1) \) lens space. Take the genus one Heegaard splitting \( H_1 \cup f_1, T^2 \times I \cup f_0, H_0 \) for \( L(2,1) \) where \( f_0 : \partial H_0 \to T^2 \times \{0\} \) is the identity map and \( f_1 : \partial H_1 \to T^2 \times \{1\} \) is
\[
f_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.
\]
Let \( \ell_i \) and \( m_i \) be the longitude and meridian of \( H_i \). Let \( \ell \) and \( m \) be the longitude and meridian of \( T^2 \times I \).

The specialized skein modules of the handlebodies, \( K_{-1}(H_0) \) and \( K_{-1}(H_1) \), correspond to subvarieties of the specialized skein module of the surface, \( K_{-1}(T^2) \). When we push a framed link from \( T^2 \times I \) into one of the handlebodies, there are relations induced by the fact that \( m_i \) is trivial and \( \ell_i \simeq \ell_i m_i \) in handlebody \( H_i \).

Recall from Example 1 that \( K_{-1}(T^2) \) is generated by \( x = -\text{tr}(m), y = -\text{tr}(\ell), \) and \( z = -\text{tr}(\ell m) \). Indeed, \( K_{-1}(T^2) = \mathbb{C}[x, y, z]/I \) where \( I \) is the ideal \( I = (x^2 + y^2 + z^2 + xyz - 4) \). Since \( f_0 \) is the identity map, the relations induced by \( m_0 \simeq \ast \) and \( \ell_0 \simeq \ell_0 m_0 \) are \( x = -\text{tr}(m) = -\text{tr}(m_0) = -2 \) and \( y = -\text{tr}(\ell) = -\text{tr}(\ell_0) = -\text{tr}(\ell_0 m_0) = -\text{tr}(\ell m) = z \).

Let \( J = (x + 2, y - z) \), then \( K_{-1}(H_0) = K_{-1}(T^2)/J \).
Remark. In the preceding paragraph (and in the paragraphs below), we are abusing notation when we use \( \text{tr}(m) \), \( \text{tr}(\ell) \), etc. We are suppressing the representations \( \rho : \pi_1(H_1) \to SL_2(\mathbb{C}) \) and \( \tilde{\rho} : \pi_1(T^2) \to SL_2(\mathbb{C}) \). The relations on the traces of the matrices come from the curves themselves. Thus it seems more instructive to emphasize the curves over the matrices.

The inverse of the gluing map for \( H_1 \) is

\[
f_1^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.
\]

Thus the inclusion of \( T^2 \) into \( H_1 \) sends \( \ell^2m \) to \( m_1 \), \( \ell \) to \( \ell_1^{-1}m_1 \), and \( \ell m \) to \( \ell_1 \). The relation induced by \( m_1 \simeq * \) uses \( f_1^{-1} \). In particular, \( m_1 \simeq * \) induces \( \text{tr}(\ell^2m) = \text{tr}(m_1) = 2 \). Using the trace identity for \( \text{SL}_2(\mathbb{C}) \) we have \( \text{tr}(\ell^2m) = \text{tr}(\ell) \ast \text{tr}(\ell m) - \text{tr}(m) \). Thus \( \text{tr}(\ell^2m) = 2 \) becomes \( yz + x = 2 \). Similarly a relation is induced by \( \ell_1 \simeq \ell_1m_1 \simeq \ell_1^{-1}m_1 \) using \( f_1^{-1} \). In particular,

\[
y = -\text{tr}(\ell) = -\text{tr}(\ell_1^{-1}m_1) = -\text{tr}(\ell_1) = -\text{tr}(\ell m) = z.
\]

Let \( K = (yz + x - 2, y - z) \), then \( K_{-1}(H_1) = K_{-1}(T^2)/K \).

Lemma 6. As a vector space over \( \mathbb{C} \), \( \text{Tor}_{1}^{K_{-1}(T^2)}(K_{-1}(H_1), K_{-1}(H_0)) \) is spanned by the set \( \{(y - z), y(y - z)\} \).

Proof. We know that \( \text{Tor}_{1}^{K_{-1}(T^2)}(K_{-1}(H_1), K_{-1}(H_0)) = (J \cap K)/(J K) \) with \( J = (x + 2, y - z) \) and \( K = (yz + x - 2, y - z) \). Take \( \alpha(x, y, z) \in (J \cap K) \). We know that

\[
\alpha(x, y, z) = p_1(x, y, z)(x + 2) + p_2(x, y, z)(y - z)
\]

and

\[
\alpha(x, y, z) = q_1(x, y, z)(yz + x - 2) + q_2(x, y, z)(y - z).
\]

Use \( (x + 2)(yz + x - 2) \in JK \) and \( (y - z)(yz + x - 2) \in JK \) to write \( q_1(x, y, z) \) as a function in \( y \). Use \( (x + 2)(y - z) \) and \( (y - z)(y - z) \) to write \( q_2(x, y, z) \) as a function in \( y \). Thus

\[
\alpha(x, y, z) = \tilde{q}_1(y)(yz + x - 2) + \tilde{q}_2(y)(y - z)
\]

in the quotient \( (J \cap K)/(JK) \). Evaluating \( \alpha(x, y, z) \) at \( (-2, y, y) \) we have

\[
0 = \alpha(-2, y, y) = \tilde{q}_1(y)(y^2 - 4) + \tilde{q}_2(y)(y - y) = \tilde{q}_1(y)(y^2 - 4).
\]

Thus \( \tilde{q}_1(y) = 0 \) and \( \alpha(x, y, z) = \tilde{q}_2(y)(y - z) \). Hence \( (J \cap K)/(JK) \) is spanned by the set \( \{(y - z), y(y - z), y^2(y - z), y^3(y - z), \ldots \} \).
Now consider the element \( y^2(y - z) - 4(y - z) \) in \((J \cap K)/(JK)\) as follows.

\[
y^2(y - z) - 4(y - z) = y^2(y - z) - 2(y - z) - 2(y - z) + (x + 2)(y - z)
\]

\[
= y^2(y - z) - y(y - z)(y - z) - 2(y - z) + x(y - z)
\]

\[
= y^2(y - z) - y^2(y - z) + yz(y - z) - 2(y - z) + x(y - z)
\]

\[
= (yz + x - 2)(y - z).
\]

Since \((yz + x - 2)(y - z) \in JK\), we have \( y^2(y - z) = 4(y - z) \) in \((J \cap K)/(JK)\). Thus \((J \cap K)/(JK)\) is spanned by the set \(\{(y - z), y(y - z)\}\).

\[\square\]

**Proposition 2.** There is no \((1 + t)\)-torsion in \(K(L(2, 1))\).

**Proof.** Any element in \((J \cap K)/(JK)\) can be written as \(\phi \otimes \alpha \otimes \phi\) where \(\alpha\) is a linear combination of \((y - z)\) and \(y(y - z)\). To show that \(\Delta^*_1(\phi \otimes \alpha \otimes \phi) = 0\) it is enough to show that \(\Delta^*_1(\phi \otimes (y - z) \otimes \phi) = 0\) and \(\Delta^*_1(\phi \otimes y(y - z) \otimes \phi) = 0\).

In \(A/(1 + t)A\) the element \(y - z\) is equal to \(y + t^3z\).

\[
\Delta^*_1(\phi \otimes (y + t^3z) \otimes \phi) = f_1^{-1}(\ell + t^3\ell m) \otimes \phi - \phi \otimes (\ell_0 + t^3\ell_0m_0)
\]

\[
= (\ell_1^{-1}m_1 + t^3\ell_1) \otimes \phi - \phi \otimes (\ell_0 + t^3\ell_0m_0)
\]

\[
= (-t^3\ell_1 + t^3\ell_1) \otimes \phi - \phi \otimes (\ell_0 + (t^3)(-t^{-3})\ell_0)
\]

\[
= 0 \otimes \phi - \phi \otimes 0
\]

\[
= 0
\]

In \(A/(1 + t)A\), the element \(y(y - z)\) is equal to \(\beta = p(t) * y^2 - \frac{1}{2}yz - \frac{1}{2}zy\) where \(p(t) = -\frac{1}{2}t^{-3} - \frac{1}{2}t^{-5}\). Apply \(\Delta^*_1\) to the element \(\phi \otimes \beta \otimes \phi\).

\[
\Delta^*_1(\phi \otimes \beta \otimes \phi) = \Delta^*_1(\phi \otimes (p(t)y^2 - \frac{1}{2}yz - \frac{1}{2}zy) \otimes \phi)
\]

\[
= f_1^{-1} \left( p(t) \ell * \ell - \frac{1}{2} \ell * \ell m - \frac{1}{2} \ell m * \ell \right) \otimes \phi
\]

\[
- \phi \otimes \left( p(t) \ell_0 * \ell_0 - \frac{1}{2} \ell_0 * \ell_0m_0 - \frac{1}{2} \ell_0m_0 * \ell_0 \right)
\]

\[
= \left( p(t)\ell_1m_1^{-1} * \ell m_1^{-1} - \frac{1}{2} \ell_1 * \ell_1m_1^{-1} - \frac{1}{2} \ell_1m_1^{-1} * \ell_1 \right) \otimes \phi
\]

\[
- \phi \otimes \left( p(t) \ell_0 * \ell_0 - \frac{1}{2} \ell_0 * \ell_0m_0 - \frac{1}{2} \ell_0m_0 * \ell_0 \right)
\]

(4)

Let \(\gamma = \ell_1m_1^{-1} * \ell_1m_1^{-1}\) as shown in Figure 6. Removing the kinks in \(\gamma\), we see that \(\gamma = t^6\delta\) where \(\delta\) is the link shown in Figure 7. Note also that \(\ell_1m_1^{-1} * \ell_1 = -t^3\delta\) and \(\ell_0m_0 * \ell_0 = -t^{-3}\delta\) where \(\delta\) is the mirror image of \(\delta\). Removing kinks and using \(\gamma\) and \(\delta\), Equation 4 becomes
\[
\Delta_1(\phi \otimes \beta \otimes \phi) = \left( p(t)t^6\delta + \frac{1}{2}t^3\ell_1 \ast \ell_1 + \frac{1}{2}t^3\delta \right) \otimes \phi
\]

(5)

\[
-\phi \otimes \left( p(t)\ell_0 \ast \ell_0 + \frac{1}{2}t^{-3}\ell_0 \ast \ell_0 + \frac{1}{2}t^{-3}\delta \right)
\]

Now apply the skein relations to \( \delta \) as shown in Figure 8. In handlebody \( H_1 \) we have \( \delta = t^2\ell_1 \ast \ell_1 + (t^{-4} - 1)[2]\phi \) and in handlebody \( H_0 \) we have \( \bar{\delta} = t^{-2}\ell_0 \ast \ell_0 + (t^4 - 1)[2]\phi \).

\[
\begin{align*}
\begin{tikzpicture}
\end{tikzpicture} & = t \begin{tikzpicture}
\end{tikzpicture} + t^{-1} \begin{tikzpicture}
\end{tikzpicture} \\
= t^2 \begin{tikzpicture}
\end{tikzpicture} & + \begin{tikzpicture}
\end{tikzpicture} - t^{-4} \begin{tikzpicture}
\end{tikzpicture} \\
= t^2 \begin{tikzpicture}
\end{tikzpicture} & + (t^{-4} - 1)[2] \begin{tikzpicture}
\end{tikzpicture}
\end{align*}
\]

Figure 8. The skein relations applied to the link \( \delta \)
Let $\beta_1 = p(t)t^6\delta + \frac{3}{2}t^3\ell_1 \ast \ell_1 + \frac{1}{2}t^3\delta$ and $\beta_0 = p(t)\ell_0 \ast \ell_0 + \frac{1}{2}t^{-3}\ell_0 \ast \ell_0 + \frac{1}{2}t^{-3}\delta$. Then $\Delta(\phi \otimes \beta \otimes \phi) = \beta_1 \otimes \phi - \phi \otimes \beta_0$. Consider

$$
\beta_1 = p(t)\left(t^8\ell_1 \ast \ell_1 + t^6(t^{-4} - 1)[2]\phi\right) + \frac{1}{2}t^3\ell_1 \ast \ell_1 + \frac{1}{2}t^5\ell_1 \ast \ell_1 + \frac{1}{2}t^3(t^{-4} - 1)[2]\phi
$$

$$
= (-\frac{1}{2}t^{-3} - \frac{1}{2}t^{-5})t^6(t^{-4} - 1)[2]\phi + \frac{1}{2}t^3(t^{-4} - 1)[2]\phi
$$

and

$$
\beta_0 = p(t)\ell_0 \ast \ell_0 + \frac{1}{2}t^{-3}\ell_0 \ast \ell_0 + \frac{1}{2}t^{-3}\left(t^{-2}\ell_0 \ast \ell_0 + (t^4 - 1)[2]\phi\right)
$$

$$
= \frac{1}{2}t^{-3}(t^4 - 1)[2]\phi.
$$

Then

$$
\Delta_1^*(\phi \otimes \beta \otimes \phi) = \beta_1 \otimes \phi - \phi \otimes \beta_0
$$

$$
= -\frac{1}{2}t(t^{-4} - 1)[2]\phi \otimes \phi - \phi \otimes \frac{1}{2}t^{-3}(t^4 - 1)[2]\phi
$$

$$
= (-\frac{1}{2}t^{-3} + \frac{1}{2}t)[2]\phi \otimes \phi - \phi \otimes (\frac{1}{2}t - \frac{1}{2}t^{-3})[2]\phi
$$

$$
= 0.
$$

Therefore the $\Delta_1^*$ maps are all zero maps and by Theorem 1 there is no $(1 + t)$-torsion in $K(L(2, 1))$. □

7. TORSION IN AN HOMOLOGY SPHERE

The computational methods detailed in the previous section are cumbersome. Indeed, the description of the skein modules of the lens spaces given by Hoste and Przytycki in [7, 8] is cleaner. However, the framework of Hochschild homology given by the current paper will hopefully allow us to use advanced ideas from homological algebra and representation theory to search for torsion in the skein module of a manifold. In particular, we hope to use the results of Serre in [15] and of Goldman and Millson in [6] to prove the following rather ambitious conjecture.

**Conjecture 2.** If $M$ is an homology sphere, then there is no $(1 + t)$-torsion in $K(M)$.

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